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ON CR-SUBMANIFOLDS OF HERMITIAN MANIFOLDS

BY

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ABSTRACT

In this paper we consider a CR-submanifold of a Hermitian manifold and prove various integrability theorems on the submanifold. When the ambient space is Kaehlerian a number of differential geometric results are also obtained.

1. Introduction

Let M be an almost Hermitian manifold[†] with almost complex structure J and Hermitian metric g and N a Riemannian submanifold immersed in M. At each point $p \in N$ let $\mathcal{D}_p$ be the maximal holomorphic subspace of the tangent space T_pN, i.e. $J\mathcal{D}_p = \mathcal{D}_p$. If the dimension of $\mathcal{D}_p$ is the same for all $p \in N$, we have a holomorphic distribution $\mathcal{D}$ on N.

Recently in [1] A. Bejancu introduced the notion of a CR-submanifold of M. Precisely, N is a CR-*submanifold* of the almost Hermitian manifold M if there exists on N a C^∞ holomorphic distribution $\mathcal{D}$ which is non-trivial ($\mathcal{D}_p \neq \{0\}$ or T_pN) such that its orthogonal complement $\mathcal{D}^\perp$ is totally real in M [2], i.e. $J\mathcal{D}_p^\perp \subset T_p^\perp N$, $T_p^\perp N$ being the normal space at p. Clearly every real hypersurface of an almost Hermitian manifold is a CR-submanifold if $\dim N > 1$.

In the present paper we show that a CR-submanifold N of a Hermitian manifold is a CR-manifold in the usual sense [4] and prove other integrability theorems on N. We then give a characterization of CR-submanifolds of complex space forms in terms of the restriction of the curvature. Umbilical and totally geodesic CR-submanifolds of Kaehler manifolds are studied in detail.

2. Preliminaries

Let N be a Riemannian submanifold in a Hermitian manifold M. We denote by ∇ (respectively, $\tilde{\nabla}$) covariant differentiation with respect to the metric on N (respectively, on M). The curvature tensor R of ∇ is given by

[†]All manifolds and their structures are assumed to be C^∞.

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$$R(X,Y)=\nabla_X\nabla_Y-\nabla_Y\nabla_X-\nabla_{[X,Y]} \tag{2.1}$$

and we denote by $\tilde{R}$ the curvature tensor with respect to $\tilde{\nabla}$. The second fundamental form σ is given by

$$\sigma(X,Y)=\tilde{\nabla}_XY-\nabla_XY, \tag{2.2}$$

for vector fields X, Y tangent to N. For a normal vector field ξ on N, we write

$$\tilde{\nabla}_X\xi=-A_\xi X+\nabla^\perp_X\xi \tag{2.3}$$

where $-A_\xi X$ (respectively, $\nabla^\perp_X\xi$) is the tangential (respectively, normal) component of $\tilde{\nabla}_X\xi$. We have

$$g(\sigma(X,Y),\xi)=g(A_\xi X,Y). \tag{2.4}$$

A normal vector field ξ is said to be parallel if $\nabla^\perp\xi=0$. The mean-curvature vector H is defined by $H=\operatorname{trace}\sigma/n$. A submanifold N is *totally umbilical* if

$$\sigma(X,Y)=g(X,Y)H. \tag{2.5}$$

If $\sigma=0$, N is said to be *totally geodesic*.

For the second fundamental form σ, we define the covariant differentiation $\bar{\nabla}$ with respect to the connection in $(TN)\oplus(T^\perp N)$ by

$$(\bar{\nabla}_X\sigma)(Y,Z)=\nabla^\perp_X(\sigma(Y,Z))-\sigma(\nabla_XY,Z)-\sigma(Y,\nabla_XZ), \tag{2.6}$$

for all X, Y, Z tangent to N. The equations of Gauss, and Codazzi are then given respectively by

$$\begin{aligned}R(X,Y;Z,W)=\tilde{R}(X,Y;Z,W)&+g(\sigma(X,W),\sigma(Y,Z))\\&-g(\sigma(X,Z),\sigma(Y,W)),\end{aligned} \tag{2.7}$$

$$(\tilde{R}(X,Y)Z)^\perp=(\bar{\nabla}_X\sigma)(Y,Z)-(\bar{\nabla}_Y\sigma)(X,Z) \tag{2.8}$$

where $R(X,Y;Z,W)=g(R(X,Y)Z,W)$ and $\perp$ in (2.8) denotes the normal component.

A Kaehler manifold M is a complex space form if it is of constant holomorphic sectional curvature. Let $M(c)$ denote a complex space form of constant holomorphic sectional curvature c. Then we have

$$\begin{aligned}\tilde{R}(\tilde{X},\tilde{Y})\tilde{Z}=\frac{c}{4}\{g(\tilde{Y},\tilde{Z})\tilde{X}&-g(\tilde{X},\tilde{Z})\tilde{Y}+g(J\tilde{Y},\tilde{Z})J\tilde{X}\\&-g(J\tilde{X},\tilde{Z})J\tilde{Y}+2g(\tilde{X},J\tilde{Y})J\tilde{Z}\},\end{aligned} \tag{2.9}$$

where $\tilde{X}$, $\tilde{Y}$ and $\tilde{Z}$ are vector fields tangent to $M(c)$.

For a submanifold N in a Hermitian manifold M, let $\mathcal{D}_p$ denote the maximal holomorphic subspace of T_pN. The distribution $\mathcal{D} : p \to \mathcal{D}_p$, $p \in N$ is called the *holomorphic distribution* of N.

3. Integrability of the holomorphic distribution

We first prove the following theorem which does not require that the submanifold be a CR-submanifold but only that the dimension of the maximal holomorphic subspace be constant.

THEOREM 3.1. *Let N be a submanifold of a Kaehler manifold M and $\mathcal{D}_p$ the maximal holomorphic subspace of T_pN. Suppose* $\dim \mathcal{D}_p = \text{const}$. *Then the holomorphic distribution $\mathcal{D}$ is integrable if and only if the second fundamental form σ satisfies $\sigma(X, JY) = \sigma(JX, Y)$ for all vector fields X and Y belonging to $\mathcal{D}$.*

PROOF. If $\mathcal{D}$ is integrable let N' be an integral submanifold, σ' the second fundamental form of N' in N and $\bar{\sigma}$ the second fundamental form of N' in M. Since $\mathcal{D}$ is holomorphic, N' is a Kaehler submanifold of M and hence $\bar{\sigma}(X, JY) = \bar{\sigma}(JX, Y)$. Now $\bar{\sigma} = \sigma + \sigma'$ and hence

$$\sigma(X, JY) - \sigma(JX, Y) = \sigma'(JX, Y) - \sigma'(X, JY), \tag{3.1}$$

but the left hand side is normal to N in M and the right hand side is tangent to N (normal to N' in N). Therefore both sides of equation (3.1) vanish giving the desired condition.

Conversely, since J is parallel with respect to $\tilde{\nabla}$,

$$0 = \sigma(X, JY) - \sigma(JX, Y) = J\tilde{\nabla}_X Y - \nabla_X JY - J\tilde{\nabla}_Y X + \nabla_Y JX$$
$$= J[X, Y] - \nabla_X JY + \nabla_Y JX.$$

Therefore J applied to the tangent vector field $[X, Y]$ is tangent to N and hence $[X, Y]$ belongs to the distribution $\mathcal{D}$.

If N is a CR-submanifold, the integrability condition of the holomorphic distribution $\mathcal{D}$ can be replaced by a weaker condition as follows.

PROPOSITION 3.2. *Let N be a CR-submanifold of a Kaehler manifold M. Then the holomorphic distribution $\mathcal{D}$ is integrable if and only if*

$$g(\sigma(X, JY), \xi) = g(\sigma(JX, Y), \xi) \tag{3.2}$$

for all $X, Y \in \mathcal{D}$ and $\xi \in J\mathcal{D}^{\perp}$.

PROOF. For any Kaehler manifold M, we have $\tilde{\nabla} J = 0$. If N is a CR-submanifold in M, (2.2) and (2.3) imply

$$J\nabla_X Z + J\sigma(X, Z) = -A_{JZ}X + \nabla_X^{\perp} JZ \tag{3.3}$$

for $X \in TN$ and $Z \in \mathscr{D}^{\perp}$. From (2.4) we get

$$g(\nabla_X Z, Y) = -g(\sigma(X, JY), JZ) \tag{3.4}$$

for $X \in TN$, $Y \in \mathscr{D}$ and $Z \in \mathscr{D}^{\perp}$. Then (3.4) gives

$$g(Z, \nabla_X Y) = g(\sigma(X, JY), JZ).$$

From this we have for $X, Y \in \mathscr{D}$ and $Z \in \mathscr{D}^{\perp}$,

$$g(Z, [X, Y]) = g(\sigma(X, JY) - \sigma(JX, Y), JZ),$$

giving the proposition.

4. CR-manifolds

We first recall the notion of a CR-manifold. Let N be a differentiable manifold and $T_C N$ its complexified tangent bundle. A CR-*structure* [4] on N is a complex subbundle $\mathscr{H}$ of $T_C N$ such that $\mathscr{H}_p \cap \bar{\mathscr{H}}_p = 0$ and $\mathscr{H}$ is involutive, i.e. for complex vector fields X and Y in $\mathscr{H}$, $[X, Y]$ is in $\mathscr{H}$. It is well known that on a CR-manifold there exists a (real) distribution $\mathscr{D}$ and a field of endomorphisms $\mathscr{J} : \mathscr{D} \to \mathscr{D}$ such that $\mathscr{J}^2 = -I_{\mathscr{D}}$. $\mathscr{D}$ is just $\mathrm{Re}(\mathscr{H} \oplus \bar{\mathscr{H}})$ and $\mathscr{H}_p = \{X - \sqrt{-1}\mathscr{J}X \mid X \in \mathscr{D}_p\}$. We can now state the following result which justifies the name of CR-submanifolds.

THEOREM 4.1. *Let M be an Hermitian manifold and N a CR-submanifold, then N is a CR-manifold.*

On N we denote by P the projection map of TN to $\mathscr{D}$ and by Q the projection to $\mathscr{D}^{\perp}$. We define a tensor field $\mathscr{J}$ of type (1, 1) on N by $\mathscr{J} = JP$. Now on N we define a complex subbundle $\mathscr{H}$ by $\mathscr{H}_p = \{X - \sqrt{-1}\mathscr{J}X \mid X \in \mathscr{D}_p\}$. The following lemma is clear.

LEMMA 4.2. $-\mathscr{J}(X - \sqrt{-1}\mathscr{J}X) \in \mathscr{H}_p$ *for every* $X \in T_p N$.

LEMMA 4.3. *For vector fields X and Y belonging to $\mathscr{D}$,*

$$Q([JX, Y] + [X, JY]) = 0.$$

PROOF. Since M is Hermitian, the Nijenhuis torsion $[J, J]$ of J vanishes. Therefore

$$0 = [J, J](JX, Y) = -[JX, Y] - [X, JY] + J[X, Y] - J[JX, JY],$$

but $[X, Y]$ and $[JX, JY]$ are tangent to N and hence $J[X, Y] - J[JX, JY]$ has no component in $\mathcal{D}^\perp$. Thus $[JX, Y] + [X, JY]$ has no $\mathcal{D}^\perp$ component.

PROOF OF THEOREM 4.1. Let X and Y be vector fields belonging to $\mathcal{D}$. Then using $[J, J] = 0$

$$\begin{aligned}
&[X - \sqrt{-1}\mathcal{J}X, Y - \sqrt{-1}\mathcal{J}Y] \\
&= [X, Y] - [JX, JY] - \sqrt{-1}[JX, Y] - \sqrt{-1}[X, JY] \\
&= -J[JX, Y] - J[X, JY] - \sqrt{-1}[JX, Y] - \sqrt{-1}[X, JY] \\
&= -\mathcal{J}[JX, Y] - JQ[JX, Y] - \mathcal{J}[X, JY] - JQ[X, JY] \\
&\quad + \sqrt{-1}\mathcal{J}^2[JX, Y] - \sqrt{-1}Q[JX, Y] + \sqrt{-1}\mathcal{J}^2[X, JY] - \sqrt{-1}Q[X, JY] \\
&= -\mathcal{J}([JX, Y] - \sqrt{-1}\mathcal{J}[JX, Y]) - \mathcal{J}([X, JY] - \sqrt{-1}\mathcal{J}[X, JY]) \\
&\quad - JQ([JX, Y] + [X, JY]) - \sqrt{-1}Q([JX, Y] + [X, JY]).
\end{aligned}$$

By Lemma 4.3 the last two terms vanish and by Lemma 4.2 the first two terms belong to $\mathcal{H}$.

5. An analytic obstruction to CR-submanifolds

Again let us suppose M is Hermitian and let Ω be the fundamental 2-form of M, i.e. $\Omega(X, Y) = g(X, JY)$. It is well known that M is Kaehlerian if and only if $d\Omega = 0$. Here however let us consider a slightly larger class of Hermitian manifolds, namely those for which $d\Omega = \Omega \wedge \omega$ for some 1-form ω called the *Lee form*. When ω is closed I. Vaisman [6] calls these manifolds *locally conformal symplectic* and they include the well known Hopf manifolds.

THEOREM 5.1. *Let M be a Hermitian manifold with $d\Omega = \Omega \wedge \omega$. Then in order that N be a CR-submanifold it is necessary that $\mathcal{D}^\perp$ be integrable.*

PROOF. Let X be a vector field in $\mathscr{D}$ and Z and W vector fields in $\mathscr{D}^\perp$. Then $\Omega(X,Z)=0$ and $\Omega(Z,W)=0$. Consequently $\Omega\wedge\omega(X,Z,W)=0$ and hence

$$\begin{aligned} 0 &= 3d\Omega(X,Z,W)\\ &= X\Omega(Z,W)-Z\Omega(X,W)+W\Omega(X,Z)\\ &\quad -\Omega([X,Z],W)-\Omega([W,X],Z)-\Omega([Z,W],X)\\ &= -g([Z,W],JX), \end{aligned}$$

but X and hence JX is arbitrary in $\mathscr{D}$ and $[Z,W]$ is tangent to N, therefore $[Z,W]$ is in $\mathscr{D}^\perp$.

6. Characterization of CR-submanifolds

We first give the following characterization of CR-submanifolds in a complex space form in terms of the curvature tensor of the ambient space.

THEOREM 6.1. *Let N be a submanifold of a complex space form $M(c)$ with $c\neq 0$. Then N is a CR-submanifold if and only if the maximal holomorphic subspaces $\mathscr{D}_p=T_pN\cap J(T_pN)$, $p\in N$, define a nontrivial differentiable distribution $\mathscr{D}$ on N such that*

$$\tilde{R}(\mathscr{D},\mathscr{D};\mathscr{D}^\perp,\mathscr{D}^\perp)=0, \tag{6.1}$$

where $\mathscr{D}^\perp$ denotes the orthogonally complementary distribution of $\mathscr{D}$ in N.

PROOF. Let X,Y,Z be vector fields tangent to N. If N is a CR-submanifold for any $X,Y\in\mathscr{D}$, and $Z\in\mathscr{D}^\perp$, (2.9) gives

$$\tilde{R}(X,Y)Z=\frac{c}{2}g(X,JY)JZ.$$

Because JZ is normal to N for any $Z\in\mathscr{D}^\perp$, we then obtain (6.1).

Conversely, if the maximal holomorphic subspaces $\mathscr{D}_p\equiv T_pN\cap J(T_pN)$, $p\in N$, define a nontrivial differentiable distribution $\mathscr{D}$ such that (6.1) holds, then (2.9) implies that

$$0=\tilde{R}(JX,X;Z,W)=\frac{c}{2}g(X,X)g(JZ,W)$$

for all $X\in\mathscr{D}$ and $Z,W\in\mathscr{D}^\perp$. From this we see that $J\mathscr{D}_p^\perp$ is perpendicular to $\mathscr{D}_p^\perp$.

Since $\mathcal{D}$ is holomorphic, $J\mathcal{D}_p^\perp$ is also perpendicular to $\mathcal{D}_p$. Therefore $J\mathcal{D}_p^\perp \subset T_p^\perp N$. This shows that N is a CR-submanifold.

7. Totally umbilical submanifolds

In this section we shall study totally umbilical CR-submanifolds in detail.

If N is a totally umbilical CR-submanifold in a Kaehler manifold M, then we have

$$\sigma(X, Y) = g(X, Y)H \tag{7.1}$$

for $X, Y \in TN$. From this we find

$$g(\sigma(X, X), JW) = g(X, X)g(H, JW) \tag{7.2}$$

for $X \in TN$ and $W \in \mathcal{D}^\perp$.

From (3.3) we have

$$A_{JW}Z = A_{JZ}W \tag{7.3}$$

for $Z, W \in \mathcal{D}^\perp$. Thus for any unit vector $Z \in \mathcal{D}^\perp$ perpendicular to W, (7.2) and (7.3) give

$$g(H, JW) = g(\sigma(X, X), JW) = g(\sigma(X, W), JX) = 0;$$

this shows that H is always perpendicular to $J\mathcal{D}^\perp$. Consequently, we have the following.

LEMMA 7.1. *If N is a totally umbilical CR-submanifold of a Kaehler manifold M, then either the totally real distribution $\mathcal{D}^\perp$ is 1-dimensional or the mean-curvature vector H is perpendicular to $J\mathcal{D}^\perp$.*

A submanifold N in a Kaehler manifold M is *anti-holomorphic* if each normal space $T_p^\perp N$ is carried into the tangent space T_pN by the complex structure J of M, that is, $J(T_p^\perp N) \subset T_pN$. In fact, an anti-holomorphic submanifold N in M is nothing but a CR-submanifold with $T_pN \oplus J(T_pN) = T_pM$, $p \in N$ if $\dim N > \frac{1}{2}\dim M$. Applying Lemma 7.1 to anti-holomorphic submanifolds, we have the following.

THEOREM 7.2. *If N is a totally umbilical anti-holomorphic submanifold in any Kaehler manifold, then either N is totally geodesic or N is a real hypersurface.*

For a CR-submanifold N, a plane section $X \wedge Z$ with $X \in \mathcal{D}$ and $Z \in \mathcal{D}^{\perp}$ is called a *CR-section.* The sectional curvature $\tilde{K}(\pi)$ of a CR-section π is called a *CR-sectional curvature.*

THEOREM 7.3. *Let N be a totally umbilical CR-submanifold of any Kaehler manifold M. Then the CR-sectional curvatures of M vanish that is* $\tilde{K}(\pi)=0$ *for all CR-sections* π.

PROOF. Since N is a totally umbilical submanifold, (2.6) implies

$$(\bar{\nabla}_X\sigma)(Y,Z)=g(Y,Z)\nabla^{\perp}_X H. \tag{7.4}$$

For any $\xi \in T^{\perp}N$, the equation (2.8) of Codazzi gives

$$\tilde{R}(X,Y;Z,\xi)=g(Y,Z)g(\nabla^{\perp}_X H,\xi)-g(X,Z)g(\nabla^{\perp}_Y H,\xi).$$

In particular, for any unit vectors $X \in \mathcal{D}$ and $Z \in \mathcal{D}^{\perp}$, we have

$$\tilde{R}(X,Z;JX,JZ)=\tilde{R}(X,Z;X,Z)=0.$$

This proves the theorem.

8. Totally geodesic submanifolds

Let M be a Kaehler manifold and N a totally geodesic CR-submanifold. By Theorem 3.1 or Theorem 3.2 the holomorphic distribution $\mathcal{D}$ of N is integrable and let N' be an integral submanifold of $\mathcal{D}$. Equation (3.4) then gives the following lemma.

LEMMA 8.1. *N' is totally geodesic in N.*

Similarly by Theorem 5.1, $\mathcal{D}^{\perp}$ is integrable and let L be an integral submanifold.

LEMMA 8.2. *L is totally geodesic in N.*

PROOF. Let ∇' be the induced connection on L. We denote by σ' the second fundamental form of L in N. Then since J is parallel with respect to $\tilde{\nabla}$, the Gauss–Weingarten equations give for Z and W tangent to L,

$$J\tilde{\nabla}_Z W = J\nabla'_Z W + J\sigma'(Z,W)$$

and

$$J\tilde{\nabla}_Z W = \tilde{\nabla}_Z JW = \nabla_Z^{\perp} JW.$$

Now $\nabla_Z^{\perp} JW$ lies in the normal bundle $T^{\perp}N$, $J\nabla_Z' W$ lies in $J\mathscr{D}' \subset T^{\perp}N$ and $J\sigma'(Z, W)$ lies in $\mathscr{D}$, since $\sigma'(Z, W)$ lies in $\mathscr{D}$. Consequently $\sigma'(Z, W) = 0$ as desired.

THEOREM 8.3. *Let N be a totally geodesic CR-submanifold of a Kaehler manifold M. Then N is locally the Riemannian product of a Kaehler submanifold and a totally real submanifold.*

PROOF. By Theorem 3.1 $\mathscr{D}$ is integrable and by Theorem 5.1 $\mathscr{D}^{\perp}$ is integrable. Thus, N is locally the product of a Kaehler submanifold and a totally real submanifold. Hence, it remains only to show that this locally product structure is Riemannian. For this it suffices to show that the projection map P (or Q) is parallel with respect to ∇.

By Lemma 8.1 integral submanifolds of $\mathscr{D}$ are totally geodesic in M and hence in N. Thus for X and Y in $\mathscr{D}$,

$$(\nabla_X P)Y = \nabla_X PY - P\nabla_X Y = \nabla_X Y - \nabla_X Y = 0.$$

For X in $\mathscr{D}$ and Z in $\mathscr{D}^{\perp}$

$$(\nabla_X P)Z = -P\nabla_X Z = -P\nabla_X'^{\perp} Z = 0$$

where $\nabla'^{\perp}$ is the connection in normal bundle of the foliation by $\mathscr{D}$ in N. Similarly by Lemma 8.2 integral submanifolds of $\mathscr{D}^{\perp}$ are totally geodesic in N. Thus

$$(\nabla_Z P)X = \nabla_Z X - P\nabla_Z X = 0.$$

Finally for Z and W in $\mathscr{D}^{\perp}$, $\nabla_Z W$ is in $\mathscr{D}^{\perp}$ and hence

$$(\nabla_Z P)W = -P\nabla_Z W = 0.$$

9. A counterexample

The purpose of this section is to give an example of a CR-submanifold of a Hermitian manifold on which $\mathscr{D}^{\perp}$ is not integrable. The example makes use of the geometry of the tangent bundle of a Riemannian manifold.

Let $\mathring{\nabla}$ be a symmetric connection on a differentiable manifold M and X a vector field on M. Then $\mathring{\nabla}$ determines a vector field X^H on TM called the *horizontal lift* of X (see e.g. [7]). On the other hand the *vertical lift* X^V of X is independent of the connection and is simply defined by $X^V\omega = \omega(X)\circ\pi$, $\pi : TM \to M$ being the natural projection and ω a 1-form on M. For a tensor field ψ of type (1, 1) on M its *horizontal lift* ψ^H may be defined by $\psi^H X^V = (\psi X)^V$ and $\psi^H X^H = (\psi X)^H$.

Recall the connection map $K : TTM \to TM$ given by $K(X^V_Z) = X_{\pi(Z)}$, $KX^H = 0$ [3]. If now G is a Riemannian metric on M and $\mathring{\nabla}$ its Levi–Civita connection, we define the *Sasaki metric* g on TM by

$$g(X, Y) = G(\pi_* X, \pi_* Y) + G(KX, KY)$$

where here X and Y are vectors on TM [3, 5]. The Levi–Civita connection ∇ of g is given in terms of $\mathring{\nabla}$ and its curvature tensor R by

$$(\nabla_{X^H} Y^H)_Z = (\mathring{\nabla}_X Y)^H_Z - \tfrac{1}{2}(R(X, Y)Z)^V,$$

$$(\nabla_{X^H} Y^V)_Z = (\mathring{\nabla}_X Y)^V_Z - \tfrac{1}{2}(R(Y, Z)X)^H,$$

$$(\nabla_{X^V} Y^H)_Z = -\tfrac{1}{2}(R(X, Z)Y)^H,$$

$$\nabla_{X^V} Y^V = 0.$$

Now let M be an almost Hermitian manifold with structure tensors (J, G), J^H the horizontal lift of J to TM and g the Sasaki metric on TM. Then it is easy to check that (J^H, g) is an almost Hermitian structure on TM. Moreover the Nijenhuis torsion is given by (see e.g. [7])

$$[J^H, J^H](X^V, Y^V) = 0,$$

$$[J^H, J^H](X^V, Y^H) = [J, J](X, Y)^V,$$

$$[J^H, J^H](X^H, Y^H)_Z = [J, J](X, Y)^H_Z + \{R(JX, JY)Z + JR(JX, Y)Z + JR(X, JY)Z + R(X, Y)Z\}^V$$

where as above $Z \in TM$ and R is the curvature tensor of G.

THEOREM 9.1. *Let M be a Kaehler manifold, then TM with the structure (J^H, g) is a Hermitian manifold which is Kaehlerian if and only if M is flat.*

PROOF. That $[J^H, J^H] = 0$ follows immediately from the fact that $[J, J] = 0$ and the curvature identities of a Kaehler manifold. Now using the fact that $\mathring{\nabla} J = 0$ we have

$$((\nabla_{X^H} J^H) Y^H)_Z = \tfrac{1}{2}(JR(X, Y)Z - R(X, JY)Z)^V$$

and similar expressions for the other components. Then clearly $R = 0$ implies that J^H is parallel. Conversely, if J^H is parallel, $R(X, Y)JZ = JR(X, Y)Z = R(X, JY)Z$ on M and hence

$$G(R(X, JY)Z, W) = G(R(X, Y)JZ, W) = G(R(JZ, W)X, Y)$$

$$= G(R(Z, W)JX, Y) = -G(R(Z, W)X, JY) = -G(R(X, JY)Z, W).$$

But X, JY, Z, W are arbitrary vector fields on M and hence $R = 0$.

We now take M to be complex projective space PC^n with the Fubini–Study metric and consider the real projective space $P\mathbf{R}^n$ as a totally real, totally geodesic submanifold imbedded in PC^n. Let N^{3n} be the set of all fibres of TPC^n over the points of $P\mathbf{R}^n$. By Theorem 9.1 TPC^n is a Hermitian manifold which is not Kaehlerian. Since $P\mathbf{R}^n$ is totally real in PC^n and J^H acts invariantly on the fibres of TPC^n, N^{3n} is a CR-submanifold of TPC^n. Let X and Y be tangent vector fields to $P\mathbf{R}^n$, then X^H and Y^H are in $\mathcal{D}^\perp$ on N^{3n}, but

$$[X^H, Y^H]_Z = [X, Y]^H_Z - (R(X, Y)Z)^V$$

$$= [X, Y]^H_Z - \tfrac{1}{4}(G(Y, Z)X - G(X, Z)Y + G(Z, JY)JX - G(Z, JX)JY)^V.$$

Taking X and Y orthonormal and $Z = Y_{\pi(Z)}$ we see that the vertical part does not vanish. Thus the distribution $\mathcal{D}^\perp$ on N^{3n} is not integrable.

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